Gibbs attractor: A chaotic nearly Hamiltonian system, driven by external harmonic force

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A chaotic autonomous Hamiltonian system, perturbed by small damping and small external force, harmonically dependent on time, can acquire a strange attractor with properties similar to that of the canonical distribution—the Gibbs attractor. The evolution of the energy in such systems can be described as the energy diffusion. For the nonlinear Pullen-Edmonds oscillator with two degrees of freedom, the properties of the Gibbs attractor and their dependence on parameters of the perturbation are studied both analytically and numerically.

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I. INTRODUCTION

The Brownian motion of a particle of mass m in the static potential $U(\mathbf{r})$ can be described by the system of Langevin equations

$$m\ddot{\mathbf{r}}+\gamma\dot{\mathbf{r}}+\nabla U(\mathbf{r})=\mathbf{f}(t), \tag{1}$$

where γ is the viscous damping parameter and $\mathbf{f}(t)$ is the random force (white noise) with Gaussian distribution and mean values

$$\langle f_i(t)\rangle=0, \quad \langle f_i(t),f_j(t')\rangle=2D\delta_{ij}\delta(t-t'); \tag{2}$$

here δ_{ij} is the Kronecker symbol and $\delta(x)$ is the Dirac delta function [1]. The damping term and the random force provide a phenomenological description of the interaction of the particle (in the potential) with its environment, which is frequently called a heat bath. The dynamical system determined by Eq. (1) with $\gamma=0$ and $D=0$ will be called isolated.

The ensemble of Brownian particles eventually will come to an equilibrium with the stationary probability distribution in the phase space given by the equation

$$Q(\mathbf{p},\mathbf{r})=N\exp\left(-\frac{H(\mathbf{p},\mathbf{r})}{\Theta}\right), \tag{3}$$

where $\mathbf{p}=m\dot{\mathbf{r}}$ is the particle momentum, $H(\mathbf{p},\mathbf{r})=\mathbf{p}^2/2m+U(\mathbf{r})$ is the Hamiltonian function of the isolated system, $\Theta=D/\gamma$ is the temperature, and N is the normalization constant. The distribution Eq. (3) is known as the canonical, or Gibbs, distribution; it serves as a central point of equilibrium statistical physics [2]. From Eq. (3) follows the equilibrium energy distribution

$$Q(E)=N\Phi(E)\exp\left(-\frac{E}{\Theta}\right), \tag{4}$$

where

$$\Phi(E)=\int \delta(E-H(\mathbf{p},\mathbf{r}))d\mathbf{p}\,d\mathbf{r} \tag{5}$$

is the energy density of the phase volume on the energy surface $H(\mathbf{p},\mathbf{r})=E$.

The canonical distribution Eq. (3) holds for the equilibrium state of the system Eq. (1) irrespective of the dimensionality of the configuration space d and regardless of the nature of the motion of the isolated system, be it regular (periodic, quasiperiodic) or chaotic.

Let us replace the random force $\mathbf{f}(t)$ by a regular one that depends on time harmonically; the equations of motion will take the form

$$m\ddot{\mathbf{r}}+\gamma\dot{\mathbf{r}}+\nabla U(\mathbf{r})=\mathbf{F}\sin\omega t. \tag{6}$$

If the motion of the isolated system is strongly chaotic, that is, nearly ergodic on the energy surfaces in a wide range of energy values, then we can expect that the motion of a weakly perturbed system (γ and F are small) will be chaotic too. In parallel with the physical picture of the Brownian motion given above, the external force $\mathbf{F}\sin\omega t$ will slowly change the energy of the system, whereas the damping γ will provide a sink for the excessive energy, thus creating the possibility of equilibrium.

The dissipative nonautonomous system Eq. (6) may demonstrate the chaotic motion on a strange attractor, which has much in common with the canonical distribution Eq. (3). For this reason it will be called the Gibbs attractor.

The main purpose of this paper is to give an example of a system with the Gibbs attractor and to describe its main features. They mostly depend on the kinetics of energy exchange between the perturbation and the isolated system, which can be described as a process of energy diffusion.

The problem that we have formulated is at the crossroad of several lines of research in nonlinear dynamics and nonstationary statistical physics. First, it is linked to the theory of chaos in nonautonomous Hamiltonian systems. In this theory the concept of energy (or action) diffusion is used for description of the infinite chaotic motion above the threshold of its onset—in models like the periodically kicked rotor [3] or a hydrogen atom in a microwave field [4,5]. Unlike our problem, in these models the strong periodic perturbation is the source that thrusts chaos on the system. Second, our problem is related to the theory of energy absorption by chaotic systems with parametric modulation, which is usually developed (in both quantum and classical approaches) for

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models of billiards with varying form [6,7]. As opposed to our problem, the main concern here is the study of slow variations, whereas we are interested in a perturbation with frequencies that are comparable to the characteristic frequencies of the unperturbed systems. Furthermore, in both aforementioned theories the damping is completely neglected. Third, our problem has a relation to the actively developing theory of Brownian motion under the influence of colored noise [8–11], since one may consider the harmonic perturbation as a limiting (monochromatic) case of narrowband noise. In contrast to our problem, in this theory the unperturbed motion is mainly one dimensional and thus regular, and damping is considered strong. Last, we note a connection to the issue of the effects of weak noise and damping on Hamiltonian systems that was discussed recently in the context of the problem of decay of metastable chaotic states [12,13].

The rest of the text is organized as follows. In Sec. II the main equation for the energy diffusion is deduced in two ways. Section III consists of a description of the unperturbed model, the Pullen-Edmonds nonlinear oscillator, and the main features of its chaotic motion. Section IV comprises a description of the main properties of the Gibbs attractor in the model—its limits in the phase space, energy distribution, conditions of existence, and character of the energy correlation. Section V contains the concluding remarks.

II. ENERGY DIFFUSION

We will study the evolution of the distribution of energy values $Q(E,t)$ for a Hamiltonian chaotic system, perturbed by small damping and small harmonic force. For the derivation of the equation of evolution for $Q(E,t)$ we at first neglect the damping (we shall restore it later, in Sec. II C).

A. The quantum approach

The simplest approach is to start from the quantum model of the unperturbed system. Let us assume that at the initial moment the system is in a stationary state $|n\rangle$ with the energy E. The external harmonic force will induce transitions with absorption $(+)$ and emission $(-)$ of the quanta $\hbar\omega$. If the motion of the classical system is chaotic and the power spectrum of the active coordinate x is continuous, then for its quantum counterpart in the quasiclassical case the density of final states with allowed transitions is high, and the rate of these transitions can be described by the Fermi golden rule. With the account of dependence of matrix elements and density of states on the energy within the transition range, we can obtain for the rate of transitions the expression

$$\dot{W}_{\pm}(E)=\frac{2\pi}{\hbar^2}\frac{F^2}{4}\left[S\pm\frac{\hbar\omega}{2}\left(S'+S\frac{\rho'}{\rho}\right)\right], \tag{7}$$

where $S=S_x(E,\omega)$ is the power spectrum of the coordinate x, $\rho=\rho(E)$ is the density of states of the isolated system, both taken at the energy E, and primes mean differentiating with respect to energy [18].

The resonant absorption and emission of quanta populate narrow bands of levels, which are located on the energy scale around values $E_{\pm k}=E\pm k\hbar\omega$ with integer k. We denote the probability of finding the system in such a band of states around the energy value E as $Q(E)$. Taking account of one-photon transitions we can write the balance equation

$$\frac{dQ(E)}{dt}=-Q(E)(\dot{W}_{+}+\dot{W}_{-})+Q(E+\hbar\omega)\dot{W}_{+}+Q(E-\hbar\omega)\dot{W}_{-}. \tag{8}$$

Assuming $Q(E,t)$ to be a smooth function of E, we can expand the arguments in the second and third terms in the right-hand side (RHS) of Eq. (8) to the order of $\hbar^2$ inclusive. This yields the equation in partial derivatives that describes the energy diffusion in the Hamiltonian system under the influence of the external harmonic field:

$$\frac{\partial Q}{\partial t}+\frac{\partial}{\partial E}(AQ)-\frac{\partial}{\partial E}\left(D\frac{\partial Q}{\partial E}\right)=0. \tag{9}$$

Here the coefficients of drift A and of diffusion D are given by the expressions

$$A(E,\omega)=\frac{\pi}{2}\omega^2F^2S\frac{\rho'}{\rho},\quad D(E,\omega)=\frac{\pi}{2}\omega^2F^2S. \tag{10}$$

It is essential that in the quasiclassical case, when the density of states could be expressed by the Weyl formula

$$\rho(E)=\frac{\Phi(E)}{(2\pi\hbar)^d}, \tag{11}$$

the coefficients A and D do not depend on Planck's constant $\hbar$. The analogous derivation of the equation for the energy diffusion in a conservative one-dimensional system under the influence of white noise was used in [19].

B. The classical approach

Equation (9) does not depend on Planck's constant, but the condition of its applicability does. The Fermi golden rule is based on the assumption of the resonant character of the transitions, which is justified only when the transition rate $\dot{W}$ is much smaller than the perturbation frequency ω. Thus the diffusion coefficient must obey the strong inequality

$$D\ll\hbar^2\omega^3, \tag{12}$$

which is too restrictive for small $\hbar$.

However, we can rederive Eq. (9) classically. In the zeroth approximation we neglect the influence of the external force on the law of motion $x(t)$. Then the instantaneous rate of energy change is $\dot{E}=\dot{x}(t)F(t)$. Variation of the energy (for the time interval T), $\Delta(T)$, in this approximation vanishes on the average for symmetry reasons, $\langle\Delta(T)\rangle=0$, whereas its averaged square is

$$\langle\Delta^2(T)\rangle=\int_0^T\int_0^T B_v(t_1-t_2)F(t_1)F(t_2)dt_1dt_2, \tag{13}$$

where $B_v(\tau)$ is the correlation function of the x component of the velocity. If the unperturbed motion is ergodic, then the correlation function is determined by the microcanonical average

$$B_v(\tau)=\frac{1}{\Phi(E)}\int \dot{x}(0)\dot{x}(\tau)\,\delta(E-H(\mathbf{p},\mathbf{r}))d\mathbf{p}d\mathbf{r}, \quad (14)$$

where $x(0)$ and $x(\tau)$ are taken on the trajectory that starts at $t=0$ at the phase point $\{\mathbf{p},\,\mathbf{r}\}$. For times T much larger than the time of decay of velocity correlations τ_c, we can rewrite Eq. (13) in the form

$$\langle\Delta^2(T)\rangle=\frac{F^2}{2}\int_0^T d\theta\int_{-\infty}^{\infty}B_v(\tau)\cos(\omega\tau)d\tau. \quad (15)$$

The internal integral is proportional to the power spectrum of velocity, $S_v(E,\omega)$. Since $S_v(E,\omega)=\omega^2S_x(E,\omega)$, we obtain for the coefficient of energy diffusion the expression $D(E,\omega)=(\pi/2)\omega^2F^2S_x(E,\omega)$, which coincides with the one given above in Eq. (10).

Let us assume that in the initial state the system has the energy E_0 and denote as $Q_0(E,t)$ the energy distribution with this initial condition, $Q_0(E,0)=\delta(E-E_0)$. Equation (15) can be rewritten as

$$\langle\Delta^2(T)\rangle=2\int_0^T D(E_0)d\theta. \quad (16)$$

The distribution Q_0 spreads with time, including energy surfaces with different densities Φ and different diffusion coefficients D. In the first approximation we can take this into account by averaging these functions with the evolving distribution $Q_0(E,t)$:

$$\langle\Delta^2(T)\rangle=2\int_0^T\frac{d\theta}{\Phi(E_0)}\int dE\,Q_0(E,\theta)G(E), \quad (17)$$

where $G(E)=\Phi(E)D(E)$. In the time interval in which the distribution $Q_0(E,t)$ can be considered narrow, the function $G(E)$ can be expanded in a Taylor series to the first order in $\Delta=E-E_0$. Thus we come to the equation

$$\langle\Delta^2(T)\rangle=2\int_0^T\left(D(E_0)+\frac{G'(E_0)}{\Phi(E_0)}\langle\Delta(\theta)\rangle\right)d\theta. \quad (18)$$

If the system on the average changes its energy with a constant rate $\langle\Delta(t)\rangle=\alpha t$, then $\langle\Delta^2(t)\rangle=\alpha^2t^2+2Dt$. Substituting this expression in Eq. (18), we obtain

$$D=D(E_0),\qquad \alpha=\frac{G'(E_0)}{\Phi(E_0)}. \quad (19)$$

The average rate of the energy variation in the state with given distribution $Q(E,t)$ can be obtained from Eq. (9):

$$\langle\dot{E}\rangle=\int (A+D')Q(E,t)dE. \quad (20)$$

From Eqs. (19) and (20) for the drift coefficient, we obtain the expression $A(E,\omega)=D(E,\omega)\ [\Phi'(E)/\Phi(E)]$, which coincides with the one given above in Eq. (10), since the density of states is proportional to the density of the phase volume [see Eq. (11)].

In the classical derivation of Eq. (20) we have assumed that the spreading of the initially localized energy distribution during the correlation time τ_c is such that the corrections to the diffusion coefficient are small in comparison with its zero order value $\sqrt{D\tau_c}D'\ll D$. Estimating the derivative as $D'\sim D/E$, where E is a characteristic energy of the system, we obtain the condition of applicability of the classical equation of energy diffusion,

$$D\ll\frac{E^2}{\tau_c}, \quad (21)$$

which is much more tolerant than Eq. (12). Bridging the gap between Eq. (12) and Eq. (21) remains a challenge for the quantum theory.

C. The account of damping

Now we turn to the account of damping. For systems in which the logarithmic rate of energy damping is constant, $\dot{E}=-\gamma E$, the energy distribution can be written as

$$Q(E,t)=e^{\gamma t}\psi(Ee^{\gamma t}), \quad (22)$$

where $\psi(z)$ is an arbitrary positive integrable function. This functional form can be expressed by the equation in partial derivatives

$$\frac{\partial Q}{\partial t}=\gamma\frac{\partial}{\partial E}(EQ). \quad (23)$$

By combining Eqs. (9) and (23) we obtain the equation of energy diffusion in the perturbed chaotic system with damping:

$$\frac{\partial Q}{\partial t}-\frac{\partial}{\partial E}\left(\gamma EQ-AQ+D\frac{\partial Q}{\partial E}\right)=0. \quad (24)$$

Its stationary solution is given by the formulas

$$Q(E)=N\Phi(E)\exp[R(E,\omega)], \quad (25)$$

$$R(E,\omega)=-\gamma\int_0^E\frac{\varepsilon}{D(\varepsilon,\omega)}d\varepsilon, \quad (26)$$

where N is the normalization constant. The stationary distribution can be canonical, Eq. (6), only in the special case when the diffusion coefficient is exactly proportional to energy.

III. THE MODEL

Equations (25) and (26) can be checked by direct numerical solution of the system Eq. (6) for any given potential $U(\mathbf{r})$. The calculation of $Q(E)$ is relatively easy; however, the estimation of the integral in the RHS of Eq. (26) is much more demanding, since at every integration step in ε the power spectrum $S_x(\varepsilon,\omega)$ has to be calculated with sufficient accuracy. Instead of this computation we will proceed with analytical approaches workable with a special choice of the model.

We take the Pullen-Edmonds oscillator [14], which describes the two-dimensional motion of a particle in a quartic potential, as the model for the isolated system. The Hamiltonian of this system is

$$H=\frac{1}{2m}(p_x^2+p_y^2)+\frac{m\omega_0^2}{2}\left(x^2+y^2+\frac{x^2y^2}{\lambda^2}\right). \quad (27)$$

In the following we use the particle mass m, the frequency of small oscillations ω_0, and the nonlinearity length λ as unit scales, and write all equations in dimensionless form.

The properties of chaotic motion of the Pullen-Edmonds model are thoroughly studied [15–17]. With an increase of energy the system becomes more chaotic both in extensive [that is, characterized by the measure of the chaotic component $\mu_s(E)$ on the surface of Poincaré section] and in intensive [that is, measured by the magnitude of the Lyapunov exponent $\sigma(E)$] aspects. For values of energy $E>2.1$ the measure $\mu_s>0.5$, and chaos dominates in the phase space; for $E>5$ the chaotic motion of the system is approximately ergodic [15].

For the selected model Eqs. (6) have the form

$$\ddot{x}+\gamma\dot{x}+x(1+y^2)=F\sin\omega t,$$

$$\ddot{y}+\gamma\dot{y}+y(1+x^2)=0; \quad (28)$$

the external force is chosen to be directed along the OX axis. It may be noted that it is sufficient to couple the external force to only one dynamical variable, as opposed to the Langevin forces given by Eq. (2): in a chaotic system the interaction between vibrational modes will secure the redistribution of energy.

For a particle in a two-dimensional potential the value of $\Phi(E)$ is proportional to the area $\phi(E)$ of the region that is bound by the equipotential line $U(x,y)=E$, namely, $\Phi(E)=2\pi\phi(E)$. For the Pullen-Edmonds model

$$\phi(E)=4\int_0^{\sqrt{2E}}\sqrt{\frac{2E-x^2}{1+x^2}}\,dx. \quad (29)$$

This integral can be calculated analytically:

$$\phi(E)=4\sqrt{2E+1}\left[\mathsf{K}\left(\sqrt{\frac{2E}{2E+1}}\right)-\mathsf{E}\left(\sqrt{\frac{2E}{2E+1}}\right)\right], \quad (30)$$

where $\mathsf{K}(x)$ and $\mathsf{E}(x)$ are complete elliptic integrals of the first and second kinds, respectively. For $E\gg 1$ this function has the asymptotic form

$$\phi(E)\approx\sqrt{32E}[\ln\sqrt{32E}-1]. \quad (31)$$

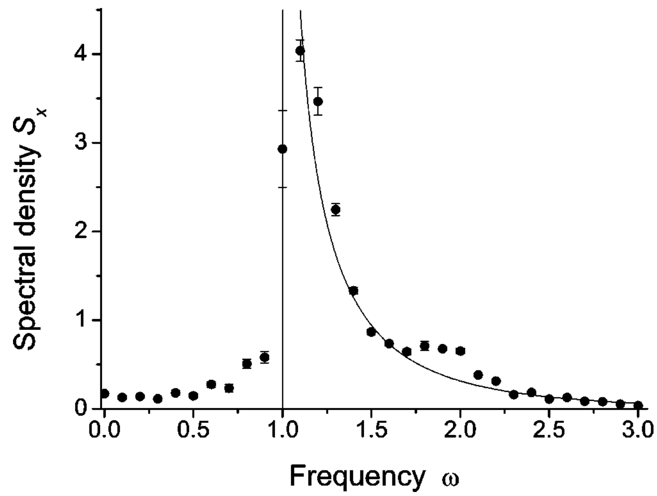

FIG. 1. The dependence of the power spectrum of the coordinate S_x of the Pullen-Edmonds model Eq. (27) on frequency ω for the energy value $E=16$. Theoretical estimate Eq. (32) (line) and numerical calculation (points).

The approximate expression for the power spectrum of the coordinate x for the Pullen-Edmonds model was obtained in Ref. [17] from the assumption of ergodicity of motion and in the ''frozen frequencies'' approximation:

$$S_x(E,\omega)=\frac{2}{3\phi(E)}(2E+1-\omega^2)^{3/2}\omega^{-2}(\omega^2-1)^{-1/2} \quad (32)$$

for $1<\omega<\sqrt{2E+1}$ and $S_x(E,\omega)=0$ outside this range. The comparison of this formula with the numerically found spectrum is shown in Fig. 1.

It can be seen that the main body of the spectrum, apart from a small interval of frequencies around $\omega=1$, is described satisfactorily.

IV. THE GIBBS ATTRACTOR

In this section we study different properties of the motion on the Gibbs attractor. For the numerical calculation we will mainly use the following set of parameters: $\gamma=2\times10^{-3}$, $F=0.3$, and $\omega=1.1$, which corresponds to $\Theta=103$. Later it will be referred to as the standard set.

(1) The motion of our model on the Gibbs attractor is chaotic: numerical computation gives for its Lyapunov exponent $\sigma=0.65(2)$. From the Kaplan-Yorke conjecture [20,21] it follows that the fractal dimensionality of the attractor D_F will differ from the dimensionality of the phase space $d_p=4$ by a quantity of the order of $\gamma/\sigma=3\times10^{-3}$. This difference is hardly noticeable: from the practical point of view the Gibbs attractor densely fills the phase space, resembling the canonical distribution.

(2) The energy distribution on the Gibbs attractor can be determined by substitution of Eq. (32) into Eq. (26); thus we get

$$R(E,\omega)=-\frac{3\gamma\sqrt{\omega^2-1}}{\pi F^2}\int^E\frac{\varepsilon\Phi(\varepsilon)}{(2\varepsilon+1-\omega^2)^{3/2}}\,d\varepsilon. \quad (33)$$

It is convenient to introduce the temperature parameter

$$\Theta = \frac{\pi F^2}{3\gamma\sqrt{\omega^2-1}}, \tag{34}$$

which is similar to the temperature of the equilibrium created by a white noise heat bath and viscous damping, $\Theta = D/\gamma$: it scales as the square of the force amplitude and as the inverse of the damping parameter.

For values of frequency $\omega \gtrsim 1$, after the substitution of the asymptotic form Eq. (32) in the numerator and disregard of the quantity (ω^2-1) in the denominator, the integration in Eq. (33) yields the simple formula

$$R(E,\omega) = -\frac{E}{\Theta}(\ln E + \ln 32 - 3). \tag{35}$$

In the derivation of Eq. (24) we have assumed that the energy damping in the absence of the external force is governed by the equation $\dot{E} = -\gamma E$. For the Pullen-Edmonds model this relation must be corrected. The exact equation for the energy damping has the form $\dot{E} = -2\gamma T$, where T is the kinetic energy (properly averaged), but the virial ratio $v(E) = 2T/E$ in general case depends on the energy. From the virial theorem for the homogeneous potentials of the power k [22] it follows that $v(E) = \text{const} = 2k/(k+2)$. For small energies $E \ll 1$, the quartic term in the potential $U(x,y)$ is negligible, and $v(E) \approx 1$, but in the domain $E \gg 1$, that we are interested in, this term becomes important. For a purely quartic potential we have $v(E) = 4/3$. Thus we may expect that for $E \gg 1$ in the Pullen-Edmonds model $1 < v(E) < 4/3$; a naive interpolation leads to the value $v(E) = 7/6$. The numerical calculation shows that the asymptotic value of $v(E)$ for high energies is very close indeed to this value, with accuracy not worse than 2% for $E > 20$. Therefore for very large values of Θ we can improve the expression for $R(E,\omega)$ just by multiplying it by the factor 7/6; the corrected value of the damping parameter will be denoted as $\tilde{\gamma}$.

Comparison of the theoretical form of the probability distribution $Q(E)$ (with the virial correction included) with its values found numerically is shown in Fig. 2.

The relaxation time τ_r for the diffusion process can be estimated as the ratio of the square of the characteristic width of the stationary distribution (e.g., the variance of the distribution $V_E = \langle E^2 \rangle - \langle E \rangle^2$) to the average value of the diffusion coefficient: $\tau_r = V_E / \langle D \rangle$. For the standard set of parameters, a theoretical estimate based on Eqs. (10), (31), and (32) gives $V_E = 420$, $\langle D \rangle = 0.67$, and $\tau_r = 620$, whereas the numerical calculation gives $V_E = 315$, $\langle D \rangle = 0.42$, and $\tau_r = 740$. In our numerical experiments we have used time averaging over the intervals about $1.7 \times 10^5 \approx 230\tau_r$, which ensures high accuracy of the stationary distribution.

(3) In opposition to the canonical distribution, the Gibbs attractor is limited in the phase space. From the equation for energy dissipation the average rate of energy loss due to damping in the state with the energy E is

$$P_- = \tilde{\gamma} E. \tag{36}$$

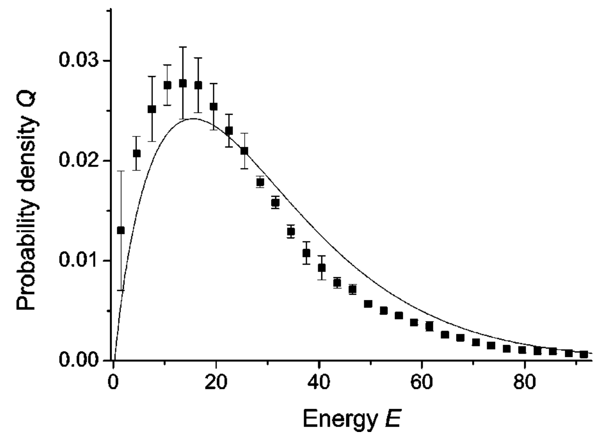

FIG. 2. The stationary probability distribution $Q(E)$ for the Gibbs attractor of the Pullen-Edmonds model with the standard set of parameters ($\gamma = 2 \times 10^{-3}$, $F = 0.3$, and $\omega = 1.1$). Theoretical form given by Eqs. (25), (31), and (35) (line) and numerical calculation (points).

The maximal rate of increase of the energy due to the influence of the external force (for $F \ll 1$) is reached when the particle moves along the OX axis with its velocity in phase with the external force, and equals

$$P_+ = \sqrt{\frac{E}{2}} F. \tag{37}$$

Balancing these quantities we find that the energy of the system cannot exceed the value

$$E_+ = \frac{F^2}{2\tilde{\gamma}^2}. \tag{38}$$

For the standard set $E_+ = 8.3 \times 10^3 = 80\Theta$. The probability of exceeding this value with the energy distribution given by Eqs. (25) and (26) is about $\exp(-750)$; thus the upper limit is not observable.

On the other hand, the attractor is not limited in the energy from below. The basic approximation of nearly ergodic motion is not applicable in this range. However, the system that at some moment is at rest, in the state with $E = 0$, can be driven by the external force to energies as high as

$$E_0 = \frac{F^2(3\omega^2+1)}{2(\omega^2-1)^2} \tag{39}$$

(in the harmonic approximation). For the standard set $E_0 = 4.72$. Thus in our case efficient communication between states with negligibly small energies and those in the nearly ergodic domain is possible.

(4) The motion of the system Eq. (28) with the standard set of parameters is found to be chaotic up to the maximal available times of numerical calculation ($t = 1.7 \times 10^5$ for ten different initial conditions). The time-averaged value $\overline{E}$ of the energy of this motion is $\overline{E} \approx 26$. Changes in parameters $\{\gamma, F, \omega\}$ that lead to diminution of the average energy of motion on the attractor, such as increase of damping or less-

ening of the amplitude of the harmonic force, or driving this force out of resonance with the chaotic unperturbed motion, eventually suppress the chaotic motion. For some initial conditions the system, after a more or less prolonged interval of transient chaotic motion with $\overline{E} \gg 1$ (we note in passing that the main results of our theory are applicable to long-living transient chaos as well as to the perpetual one) rapidly turns to regular motion on a limit cycle or a torus with $\overline{E} \lesssim 1$. This behavior can have two explanations.

(A) The Gibbs attractor turns into a semiattracting set that supports a metastable chaotic motion.

(B) The Gibbs attractor endures, but its basin of attraction shrinks, thus ceding larger parts of the phase space to basins of regular attractors.

At present we cannot choose between these alternatives.

We can establish the border of the domain of the apparent presence of the Gibbs attractor conventionally, as the surface in the parameter space at which for 80% of randomly chosen initial conditions the commencement of the regular motion takes less than 3×10^4 units of time. The detailed definition of this surface is a laborious task, so we have restricted ourselves by variation of each parameter in turn, with the other two being fixed at the standard values.

With an increase of damping the domain of presence of the Gibbs attractor is limited by the value $\gamma=5\times10^{-3}$, where the transient chaotic motion has $\overline{E}\approx 11$. With diminishing γ the chaos persists: even for $\gamma=0$ the system displays chaotic motion, accompanied by unlimited spread of the energy distribution with time.

With a decrease of the amplitude of the force, the domain of presence of the Gibbs attractor is limited by the value $F\approx 0.11$ where the transient chaotic motion has $\overline{E}\approx 6$. The increase of the force F up to values far beyond $F\approx 1$ does not bring any noticeable changes in the character of motion.

This can be explained as follows. The magnitude of the external force F must be compared to the averaged (absolute) value of the force in the same direction, created by the static potential $F_U=\langle|-\partial U/\partial x|\rangle$. On the energy surface E it can be calculated by integration:

$$F_U=\frac{4}{3\phi(E)}\int_0^{\sqrt{2E}} x\,\frac{3+2E-x^2}{1+x^2}\sqrt{\frac{2E-x^2}{1+x^2}}\,dx. \quad (40)$$

For large E this expression has the asymptotics $F_U \sim E/\ln E$. The temperature parameter can be written as $\Theta=\kappa F^2$ with $\kappa\gg 1$, and the average energy of chaotic motion in the domain of existence of the Gibbs attractor, $\Theta\gg 1$, depends on Θ as $\langle E\rangle\sim\Theta/\ln\Theta$. Thus for the average potential force we have the estimate $\langle F_U\rangle\sim\kappa^2F^2\ln^{-2}\kappa F$, and for the ratio $F/\langle F_U\rangle$ we get

$$\frac{F}{\langle F_U\rangle}\sim\frac{\ln^2\kappa F}{\kappa^2 F}. \quad (41)$$

This quantity is always small and, moreover, decreases with growth of F. In physical terms, the harmonic force, albeit large in comparison with the scales of the unperturbed model, heats the system to the level of the energy content, at which it becomes relatively small in comparison with the average potential force, thus maintaining the energy diffusion picture adequately.

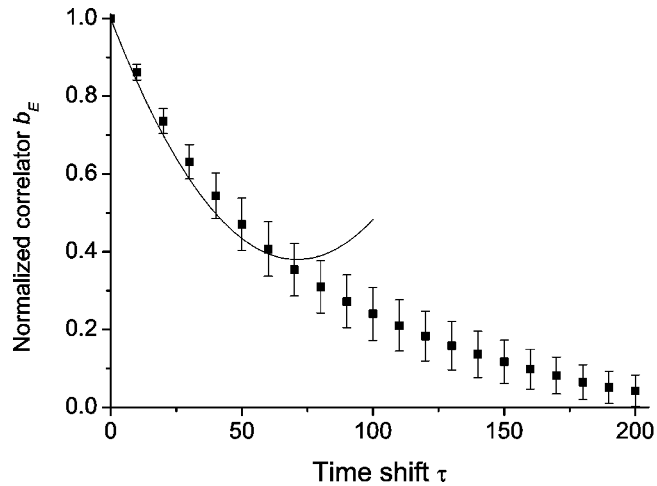

FIG. 3. The normalized correlation function of energy $b_E(\tau)$ for the motion on the Gibbs attractor of the Pullen-Edmonds model with the standard set of parameters ($\gamma=2\times10^{-3}$, $F=0.3$, and $\omega=1.1$). The time shift τ is measured in periods of the harmonic field. Theoretical approximation by first three terms of the expansion Eq. (43) (line) and numerical calculation (points).

Finally, on the frequency scale the domain of presence of the Gibbs attractor is limited by the band from $\omega\approx 0.93$ where the transient chaotic motion has $\overline{E}\approx 7$, to $\omega\approx 1.6$ where $\overline{E}\approx 11$.

(5) Since the energy of the system in our approach is the main dynamical quantity, it is appropriate to study its correlation function

$$B_E(\tau)=\langle E(0)E(\tau)\rangle-\langle E\rangle^2. \quad (42)$$

The first term in the RHS of this equality can be expanded in a Taylor series in the time shift τ and written as

$$\langle E(0)E(\tau)\rangle=\sum_{n=0}^{\infty}K_n\frac{\tau^n}{n!}, \quad (43)$$

where the coefficients K_n are equal to the products of the initial energy and of the initial value of the energy's nth time derivative, averaged with the stationary distribution,

$$K_n=\left\langle E(0)\,\frac{d^nE}{dt^n}(0)\right\rangle. \quad (44)$$

From Eq. (24) for the local rate of the energy change we have $\dot{E}=-\gamma E+A+D'$ and

$$K_1=\langle E\dot{E}\rangle=\langle-\gamma E^2+AE+D'E\rangle. \quad (45)$$

The higher time derivatives of E can be found by consequent differentiation of Eq. (24) in time, recurrent substitutions, and integrations by parts. For example, $K_2=\langle E\ddot{E}\rangle$, where

$$\ddot{E}=(\gamma E-A-D')(\gamma-A'-D'')+(A''+D''')D. \quad (46)$$

The normalized correlation function of energy $b_E(\tau) = B_E(\tau)/B_E(0)$, found in the numerical calculation, is compared to the theoretical estimate in Fig. 3.

The rate of decay of the energy correlations is determined by a competition of the absorption, which dominates in the lower energy range, and of the loss of energy through damping, which prevails in the upper range; they are balanced by the stationarity equation $\langle \dot{E} \rangle = 0$. The dependence of this rate on the parameters of the model is rather complicated.

V. CONCLUSION

The study of the system of the type Eq. (8) may be important, since for many experimentally relevant Hamiltonian models one can indicate some mechanism of relaxation, which could be approximated by the viscous damping terms at least qualitatively.

The obvious candidates for further studies of the Gibbs attractors are chaotic billiards. Our equations permit us to get some simple estimates for this case. Since a billiard *per se* has only two dimensional parameters, the particle mass m and some characteristic size a, the scale of time depends on the initial conditions and is proportional to $E^{-1/2}$. Thus the power spectrum of the coordinate can be written (with m and a as units) in the scaling form

$$S_x(E,\omega) = \frac{1}{\sqrt{E}} g\left(\frac{\omega}{\sqrt{E}}\right), \tag{47}$$

where $g(z)$ is some positive integrable function. From discontinuities of velocity for the law of motion $x(t)$ it follows that the high frequency asymptotics of the power spectrum has the form $S_x(E,\omega) \propto \omega^{-4}$. Combining this formula with Eqs. (10), (26), and (47), we obtain the approximate form of the exponent in the stationary distribution:

$$R(E,\omega) \approx -\frac{\sqrt{E}}{\Theta}, \tag{48}$$

where $\Theta = cF^2\gamma^{-1}\omega^{-2}$ and c is a numerical constant that depends on the specific form of the billiard. This estimate is valid for $E \ll \omega^2$ for billiards of any dimensionality.

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